

Reduced modular symmetries of threshold corrections and gauge coupling unification

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ABSTRACT: We revisit the question of gauge coupling unification at the string scale in orbifold compactifications of the heterotic string for the supersymmetric Standard Model. In the presence of discrete Wilson lines threshold corrections with modular symmetry that is a subgroup of the full modular group arise. We find that reduced modular symmetries not previously reported are possible. We conjecture that the effects of such threshold corrections can be simulated using sums of terms built from Dedekind eta functions to obtain the appropriate modular symmetry. For the cases of the \mathbb{Z}_8 -I orbifold and the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold it is easily possible to obtain gauge coupling unification at the “observed” scale with Kähler moduli T of approximately one.

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1 Introduction

It is well known that when the $SU(3)_{\text{colour}} \times SU(2)_L \times U(1)_Y$ gauge coupling constants $g_{1,2,3}$ are extrapolated to high energies using the renormalisation group equations of the supersymmetric Standard Model with just two Higgs doublets [1, 2] the three gauge couplings reach a common value $g(m_X) = g_{\text{string}} \simeq 0.7$ at a scale $m_X \simeq 2 \times 10^{16}$ GeV. (Arguably, this is the best evidence to date for the presence of supersymmetry underlying the Standard Model.) In contrast, in the context of heterotic string theory, unification is expected [3] at a scale $m_{\text{string}} \simeq 5.27 \times 10^{17} g_{\text{string}}$ GeV, a factor of order twenty or so larger than the “observed” unification scale. (Arguably, this is the best evidence to date that string theory is wrong!) However, the renormalisation group equations used in the extrapolation do not include contributions from the massive states associated with the unification threshold. In the case of string unification, since infinite towers of massive states contribute above the string threshold, one might hope for substantial contributions Δ_a from such loops to the renormalisation group equations for the gauge coupling constants $g_a(\mu)$, ($a = 1, 2, 3$) of the (simple) group G_a

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_{\text{string}}^2} + \frac{b_a}{16\pi^2} \ln \frac{m_{\text{string}}^2}{\mu^2} + \frac{1}{16\pi^2} \Delta_a \quad (1.1)$$

Here k_a is the affine level of the gauge group factor G_a ,

$$b_a = -3C(G_a) + \sum_{\mathbf{R}_a} h_{\mathbf{R}_a} T(\mathbf{R}_a) \quad (1.2)$$

is the one-loop beta function coefficient for G_a , with $C(G_a)$ the quadratic Casimir for the adjoint representation, $T(\mathbf{R}_a) = \text{Tr } Q_a^2$ where Q_a is any generator of G_a in the representation \mathbf{R}_a , and $h_{\mathbf{R}_a}$ is the number of chiral multiplets in the representation R_a . We use the standard normalisation of the generators such that for the fundamental representation \mathbf{N} of the group $SU(N)$, $T(\mathbf{N}) = 1/2$.

The threshold corrections Δ_a in general depend upon the various Kähler and complex structure moduli T_i and U_j associated with the 6-dimensional space on which the heterotic string is compactified, and it is conceivable that these corrections can bridge the gap between the observed unification scale m_X and the string scale m_{string} , at least for certain, hopefully realistic, values of the moduli. Of course, this is not the only approach to solving this problem. As discussed in [4], other possibilities include string GUT models, non-standard affine levels, light supersymmetry thresholds, extra non-supersymmetric matter, no-supersymmetric strings, and strong-coupling effects in string theory.

The toroidal world sheet associated with a (closed) string loop is characterised by a modular parameter τ and invariance under $PSL(2, \mathbb{Z})$ global modular transformations of the form

$$\tau \rightarrow \frac{\alpha\tau - i\beta}{i\gamma\tau + \delta} \quad (1.3)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\gamma - \beta\delta = 1$ in turn induces target-space invariance under transformations of the moduli T_k and U_k . Generically, this modular symmetry is also $PSL(2, \mathbb{Z})$

$$T_k \rightarrow \frac{a_k T_k - ib_k}{ic_k T_k + d_k} \quad (1.4)$$

with $a_k, b_k, c_k, d_k \in \mathbb{Z}$ and $a_k d_k - b_k c_k = 1$. Up to constant terms, independent of the moduli, the threshold correction arising from such an $\mathcal{N} = 2$ sector has the form

$$\Delta_a = -(b'_a{}^k - \delta_{\text{GS}}^k) \ln [(T_k + \bar{T}_k) |\eta(T_k)|^4] \quad (1.5)$$

with a similar contribution from the complex-structure modulus U_k . Here δ_{GS}^k is the (universal) coefficient of the Green-Schwarz term, and $\eta(T)$ is the Dedekind eta function

$$\eta(T) := e^{-\pi T/12} \prod_{n=1}^{\infty} (1 - e^{-2\pi n T}) \quad (1.6)$$

Note that for large values of T the eta function $\eta(T) \sim e^{-\pi T/12}$, so that $\Delta_a \propto T$. It is this feature that is exploited in attempts to bridge the gap between m_X and m_{string} . Under a modular transformation of the form given in eqn (1.4)

$$T + \bar{T} \rightarrow \frac{T + \bar{T}}{|icT + d|^2} \quad (1.7)$$

$$\eta(T) \rightarrow (icT + d)^{1/2} \eta(T) \quad (1.8)$$

up to a constant, so that Δ_a is indeed invariant.

In this paper we are concerned only with orbifold compactifications in which the 6-torus T^6 is quotiented with a \mathbb{Z}_N or $\mathbb{Z}_N \times \mathbb{Z}_M$ point group. For most of these there are

θ^p -twisted sectors of the generator(s) θ of the point group that leave a single plane invariant; such twisted sectors are referred to as $\mathcal{N} = 2$ sectors. The calculation of the contribution of a twisted sector with an invariant plane T_k^2 to the threshold corrections entails a sum over the momentum (m_{2k-1}, m_{2k}) and winding numbers (n_{2k-1}, n_{2k}) that arise in the zero-mode expansion of the string world sheet in such sectors, and it is this that generates the dependence on the moduli T_k and U_k . The foregoing result applies whenever the $\mathcal{N} = 2$ sector fixed planes are such that the torus T^6 on which the point group action is realised can be decomposed into a direct product $T^4 \oplus T^2$ with the fixed plane lying in T^2 . This is certainly *not* the case for all orbifolds. Consider, for example, the \mathbb{Z}_6 -II orbifold. It has point-group generator θ with eigenvalues $e^{2\pi i v_k}$, ($k = 1, 2, 3$) where

$$(v_1, v_2, v_3) = \frac{1}{6}(2, 1, -3) \quad (1.9)$$

The symmetry may be realised on several lattices, but in particular on the $SU(3) \oplus G_2 \oplus SO(4)$ and $SU(6) \oplus SU(2)$ lattices; in both cases the action of the point-group is given by the Casimir operators of the various groups. Evidently the θ^2, θ^4 and θ^3 -twisted sectors all have fixed planes. In the first realisation, the lattice is the product of three 2-tori $T_1^2 \oplus T_2^2 \oplus T_3^2$, with T_3^2 the fixed plane of θ^2, θ^4 , and T_1^2 the fixed plane of θ^3 . Since θ acts as \mathbb{Z}_3 on T_1^2 , the complex structure $U_1 = e^{2\pi i/3}$ is fixed and there is no associated modular symmetry. On T_3^2 , however, θ acts as a reflection \mathbb{Z}_2 and the complex structure U_3 is arbitrary. In the first case, the above analysis applies, and there is the full $PSL(2, \mathbb{Z})$ modular symmetry for the Kähler moduli T_1 and T_3 , as well as for the complex structure U_3 . (Of course, there is also a Kähler modulus T_2 associated with the second plane, but since this is not an $\mathcal{N} = 2$ plane, it does not enter the threshold corrections.) However, in the second case the lattice is $T^5 \oplus T^1$ and the previous analysis does not apply. Such cases must be treated on a case by case basis, and in this case it was shown [5] that the modular group associated with T_1 is $\Gamma^0(2)$, meaning the subgroup of $PSL(2, \mathbb{Z})$ defined in eqn (1.4) with the integer $b_1 = 0 \bmod 2$. Similarly, the modular group associated with T_3 is $\Gamma^0(3)$, corresponding to the subgroup with $b_3 = 0 \bmod 3$. The modular symmetry associated with U_3 was shown to be $\Gamma_0(3)$, corresponding to $c'_3 = 0 \bmod 3$, with c'_3 the analogue for U_3 of c_3 . As in the separable case, the threshold corrections Δ_a may be calculated and are indeed invariant under the reduced modular symmetries. For example, the T_3 - and U_3 -dependent threshold corrections are [6]

$$\begin{aligned} \Delta_a = & -2b_a^{(1, \theta^2)} \ln [(T_3 + \bar{T}_3)|\eta(T_3)|^4 (U_3 + \bar{U}_3)|\eta(U_3)|^4] \\ & -2b_a^{(1, \theta^2)} \ln \left[(T_3 + \bar{T}_3) \left| \eta\left(\frac{T_3}{3}\right) \right|^4 (U_3 + \bar{U}_3)|\eta(3U_3)|^4 \right] \end{aligned} \quad (1.10)$$

The $PSL(2, \mathbb{Z})$ modular symmetries are also broken by the presence of discrete background Wilson lines [7–11]. Besides the standard degrees of freedom X^i, Ψ^i associated with the 10-dimensional string, the heterotic string has 16 additional left-moving bosonic degrees of freedom $X^I(\tau + \sigma)$ on the $E_8 \times E_8$ lattice. Background Wilson lines A_a^I couple the X^I to the six bosonic degrees of freedom X^a associated with the (orbifold) compactified

dimensions via a term in the world-sheet action

$$S \supset \frac{1}{2\pi} \int d\tau d\sigma \epsilon^{\alpha\beta} A_a^I \partial_\alpha X^a \partial_\beta X^I \quad (1.11)$$

This has the effect of shifting the canonical left- and right-moving momenta by terms involving the quantised momentum on the $E_8 \times E_8$ lattice, but also of shifting the lattice momentum by terms involving the winding numbers. It is this mixing that modifies the modular-invariance properties. However, unlike the situation in the absence of Wilson lines, to our knowledge the string loop threshold corrections in the presence of discrete background Wilson lines have not actually been calculated¹. A systematic study of the modular symmetries in the presence of discrete Wilson lines was made in [13] for all Abelian orbifolds whose point-group action is realised by Coxeter elements or their generalisations.

In this paper we shall revisit this question, but only for those orbifolds in which the $\mathcal{N} = 2$ fixed planes are separable into the T^2 part of the 6-torus $T^4 \oplus T^2$ on which the point group is realised. We also restrict ourselves to models with *no* \mathbb{Z}_2 fixed planes; as is apparent from the previous treatment [13], such models are considerably more complicated. In the notation of that paper, only (certain compactifications of) the \mathbb{Z}_6 -I, \mathbb{Z}_8 -I, \mathbb{Z}_{12} -I, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifolds satisfy these criteria. We also speculate on the effect that our new results might have on the question of whether the threshold corrections can bridge the gap between the observed unification scale m_X of the supersymmetric Standard Model and m_{string} . In the absence of discrete Wilson lines, such unification can only be achieved with large values $T \sim 26$ of the Kähler moduli [9]. In any case, it was also shown that this unification is not possible for the \mathbb{Z}_6 -I or \mathbb{Z}_{12} -I orbifolds. Further, for the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, whose threshold corrections depend upon all three Kähler moduli $T_{1,2,3}$, it is reasonable, in our view, to insist that unification is achievable with all three moduli of comparable size $T_1 \simeq T_2 \simeq T_3$, in which case of these only the $\mathbb{Z}_3 \times \mathbb{Z}_6$ survives. Of course, one may take the opposite view to ours and make a virtue of highly anisotropic compactifications; see, for example, Hebecker & Trapletti [14].

Thus, in the end, we pursue the question of threshold corrections just for the \mathbb{Z}_8 -I and $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifolds. We shall argue that the considerably reduced modular symmetry that we now find can arise when discrete Wilson lines are present allows unification in both models with Kähler moduli $T \sim 1$ of order unity, if we conjecture that the effect of such threshold corrections can be simulated using sums of terms built from Dedekind eta functions. There have been some attempts to construct realistic models on both of these orbifolds; see, for example, Groot Nibbelink & Loukas [15] on the \mathbb{Z}_8 orbifold and Niles & Vaudrevange [16] on the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, and it will be interesting to see whether our results apply to their models.

¹The matter was addressed by Kaput & Paleani in 2010 [12], but we are unclear as to the status of this paper since it has not been published. We are grateful to the referee for drawing our attention to it.

2 Duality symmetry and Wilson lines

The action of the point group generator θ of a \mathbb{Z}_N orbifold on the defining lattice basis \mathbf{e}_a , ($a = 1, 2, \dots, 6$), is given by the elements of a matrix Q defined by

$$\theta \mathbf{e}_a = Q_{ba} \mathbf{e}_b \quad (2.1)$$

The 16×6 dimensional Wilson line matrix A with elements A_a^I may be written as

$$A := (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6) \quad (2.2)$$

where \mathbf{A}_a are 16-dimensional column vectors. It is required to satisfy [10]

$$A(1 - Q) \in \mathbb{Z} \quad (2.3)$$

and this leads to the constraints on the vectors \mathbf{A}_a that are listed in Appendix B of [13]. A must also satisfy

$$A^t C A (1 - Q) + (1 - Q^*) A^t C A \in 2\mathbb{Z} \quad (2.4)$$

where C is the Cartan matrix for $E_8 \times E_8$ and

$$Q^* := (Q^t)^{-1} \quad (2.5)$$

For the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, analogous constraints apply to the matrices Q_θ and Q_ω representing the two generators θ and ω .

As detailed in Appendix B of [13], the fixed planes of the various orbifolds may be classified according to how the point-group generator(s) acts on that plane. The only possibilities are \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_4 and \mathbb{Z}_6 . How they arise in the $T^4 \oplus T^2$ separable orbifolds is indicated in Table 1. Also displayed are the restrictions following from eqn (2.3) on the Wilson lines $(\mathbf{A}_{2k-1}, \mathbf{A}_{2k})$ in that plane. (The twists associated with the point-group generators θ (and ω) are listed in Appendix A of [13], for example.)

The modular symmetry of the the Kähler moduli T_k associated with the fixed torus

\mathbb{Z}_N fixed plane	Orbifold	Twisted sector	$(\mathbf{A}_{2k-1}, \mathbf{A}_{2k})$
\mathbb{Z}_2	\mathbb{Z}_4	θ^2	$2\mathbf{A}_3 = 0 \bmod 1 = 2\mathbf{A}_6$
	$\mathbb{Z}_6\text{-II}$	θ^2	$2\mathbf{A}_5 = 0 \bmod 1 = 2\mathbf{A}_6$
	$\mathbb{Z}_8\text{-II}$	θ^2	$2\mathbf{A}_5 = 0 \bmod 1 = 2\mathbf{A}_6$
	$\mathbb{Z}_{12}\text{-II}$	θ^2	$2\mathbf{A}_5 = 0 \bmod 1 = 2\mathbf{A}_6$
\mathbb{Z}_3	$\mathbb{Z}_6\text{-I,II}$	θ^3	$\mathbf{A}_1 - \mathbf{A}_2 = 0 \bmod 1, 3\mathbf{A}_1 = 0 \bmod 1$
	$\mathbb{Z}_{12}\text{-I}$	θ^3	$\mathbf{A}_5 - \mathbf{A}_6 = 0 \bmod 1, 3\mathbf{A}_5 = 0 \bmod 1$
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	ω^k	$\mathbf{A}_1 - \mathbf{A}_2 = 0 \bmod 1, 3\mathbf{A}_1 = 0 \bmod 1$
		θ^k	$\mathbf{A}_3 - \mathbf{A}_4 = 0 \bmod 1, 3\mathbf{A}_3 = 0 \bmod 1$
		$(\theta\omega^2)^k$	$\mathbf{A}_5 - \mathbf{A}_6 = 0 \bmod 1, 3\mathbf{A}_5 = 0 \bmod 1$
	$\mathbb{Z}_3 \times \mathbb{Z}_6$	ω^k	$\mathbf{A}_1 - \mathbf{A}_2 = 0 \bmod 1, 3\mathbf{A}_1 = 0 \bmod 1$
\mathbb{Z}_4	$\mathbb{Z}_8\text{-I}$	θ^4	$\mathbf{A}_5 = 0 \bmod 1 = 2\mathbf{A}_6$
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	ω^k	$\mathbf{A}_1 = 0 \bmod 1 = 2\mathbf{A}_2$
		θ^k	$\mathbf{A}_3 = 0 \bmod 1 = 2\mathbf{A}_4$
		$(\theta^2\omega^2)^k$	$\mathbf{A}_5 = 0 \bmod 1 = 2\mathbf{A}_6$
\mathbb{Z}_6	$\mathbb{Z}_3 \times \mathbb{Z}_6$	θ^k	$\mathbf{A}_3 = 0 \bmod 1 = \mathbf{A}_4$
		$(\theta\omega^4)^k$	$\mathbf{A}_5 = 0 \bmod 1 = \mathbf{A}_6$
	$\mathbb{Z}_6 \times \mathbb{Z}_6$	ω^k	$\mathbf{A}_1 = 0 \bmod 1 = \mathbf{A}_2$
		θ^k	$\mathbf{A}_3 = 0 \bmod 1 = \mathbf{A}_4$
		$(\theta\omega^5)^k$	$\mathbf{A}_5 = 0 \bmod 1 = \mathbf{A}_6$

Table 1. Classification of fixed planes by the action of the point group on that plane.

T_k^2 , in which T_k transforms as in eqn (1.4), is constrained [11] by

$$c_k(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) \in \mathbb{Z} \quad (2.6)$$

$$c_k \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) \in 2\mathbb{Z} \quad (2.7)$$

$$c_k(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) J \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C \in \mathbb{Z} \quad (2.8)$$

$$2(1 - d_k)(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) - c_k(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) J \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) \in 2\mathbb{Z} \quad (2.9)$$

$$2(1 - a_k)C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) - c_k C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) J \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) \in 2\mathbb{Z} \quad (2.10)$$

$$2(2 - a_k - d_k) \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) - c_k \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) J \begin{pmatrix} \mathbf{A}_{2k-1}^t \\ \mathbf{A}_{2k}^t \end{pmatrix} C(\mathbf{A}_{2k-1}, \mathbf{A}_{2k}) \in 4\mathbb{Z} \quad (2.11)$$

where

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.12)$$

We shall discuss in turn the modular symmetries of the three non- \mathbb{Z}_2 fixed plane classes displayed in Table 1.

2.1 \mathbb{Z}_6 -planes

Consider first the \mathbb{Z}_6 -plane T_2^2 arising in the θ^k -twisted sectors of the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold. Eqn (2.3) requires only that the Wilson lines in that torus are $E_8 \times E_8$ lattice vectors. They are also constrained by eqn (2.4) which in this case gives

$$\begin{pmatrix} \mathbf{A}_3^t C \mathbf{A}_3 & -\mathbf{A}_3^t C \mathbf{A}_3 - \mathbf{A}_4^t C \mathbf{A}_4 \\ \mathbf{A}_3^t C \mathbf{A}_3 + 2\mathbf{A}_3^t C \mathbf{A}_4 + \mathbf{A}_4^t C \mathbf{A}_4 & \mathbf{A}_4^t C \mathbf{A}_4 \end{pmatrix} \in 2\mathbb{Z} \quad (2.13)$$

which is automatically satisfied when \mathbf{A}_3 and \mathbf{A}_4 are lattice vectors. Eqn (2.6) is also satisfied since c_2 is an integer. However, a new constraint arises from eqn (2.7), namely that

$$c_2 \mathbf{A}_3^t C \mathbf{A}_4 \in 2\mathbb{Z} \quad (2.14)$$

Since \mathbf{A}_3 and \mathbf{A}_4 are lattice vectors, $\mathbf{A}_3^t C \mathbf{A}_4 \in \mathbb{Z}$, so if $\mathbf{A}_3^t C \mathbf{A}_4$ is even, then eqn (2.8) is satisfied without any further constraint on c_2 . Alternatively, if $\mathbf{A}_3^t C \mathbf{A}_4$ is odd, then we require that $c_2 = 0 \bmod 2$. In both cases, eqns (2.8) ... (2.10) are then satisfied without further constraint. If $\mathbf{A}_3^t C \mathbf{A}_4$ is even, eqn (2.11) is also satisfied without further constraint. When $\mathbf{A}_3^t C \mathbf{A}_4$ is odd, then since c_2 must be even it follows from the unimodularity that a_2 and d_2 are both odd, and this ensures that the first term on the left-hand side of eqn (2.11) is $0 \bmod 4$. However, ensuring that the second term is also $0 \bmod 4$ requires that $c_2 = 0 \bmod 4$. Thus there are two generic cases for the modular symmetry of T_2 . If $\mathbf{A}_3^t C \mathbf{A}_4$ is even, then it is the full $PSL(2, \mathbb{Z})$. Otherwise it is $\Gamma_0(4)$, the subgroup of $PSL(2, \mathbb{Z})$ with $c_2 = 0 \bmod 4$. This alternative was overlooked in reference [13].

The torus T_3^2 is fixed in the $(\theta\omega^4)^k$ -twisted sectors. Since the action of θ in T_3^2 is just the square of the action of ω , this too is a \mathbb{Z}_6 -plane. The action of ω in T_3^2 is the same as its action in T_2^2 , so the constraint deriving from eqn (2.4) is the same as that displayed in eqn (2.13) but with $(\mathbf{A}_5, \mathbf{A}_6)$ replacing $(\mathbf{A}_3, \mathbf{A}_4)$. It follows again that there are two generic cases for the modular symmetry of T_3 . If $\mathbf{A}_5^t C \mathbf{A}_6$ is even, then it is the full $PSL(2, \mathbb{Z})$. Otherwise it is $\Gamma_0(4)$. The same analysis applies to the three \mathbb{Z}_6 -planes of the $\mathbb{Z}_6 \times \mathbb{Z}_6$ orbifold, so the conclusion is that these are the two generic symmetries for *all* (Kähler) moduli in \mathbb{Z}_6 -planes.

2.2 \mathbb{Z}_4 -planes

For the fixed plane T_3^2 that arises in the θ^4 -twisted sector of the \mathbb{Z}_8 orbifold, the first \mathbb{Z}_4 -plane entry in Table 1, the constraint given in eqn (2.4) yields

$$\begin{pmatrix} 2\mathbf{A}_5^t C \mathbf{A}_5 & \mathbf{A}_5^t C \mathbf{A}_5 + 4\mathbf{A}_5^t C \mathbf{A}_6 + 2\mathbf{A}_6^t C \mathbf{A}_6 \\ -\mathbf{A}_5^t C \mathbf{A}_5 - 2\mathbf{A}_6^t C \mathbf{A}_6 & 2\mathbf{A}_6^t C \mathbf{A}_6 \end{pmatrix} \in 2\mathbb{Z} \quad (2.15)$$

where \mathbf{A}_5 and $2\mathbf{A}_6$ are lattice vectors. This is satisfied provided only that

$$\mathbf{A}_6^t C \mathbf{A}_6 \in \mathbb{Z} \quad (2.16)$$

If \mathbf{A}_6 is a lattice vector, then eqn (2.16) is automatically satisfied, and the analysis of the modular constraints is the same as that given above for the \mathbb{Z}_6 -planes. In this case, then, the conclusions are identical.

We therefore need only consider the alternative, namely that \mathbf{A}_6 is *not* a lattice vector. In this case, eqn (2.6) requires that $c_3 = 0 \bmod 2$. Then, the only further constraint from eqn (2.7) is that $c_3 \mathbf{A}_5^t C \mathbf{A}_6$ is even. Now, $\mathbf{A}_5^t C \mathbf{A}_6$ is either an integer or half-integer. If the former, then there is no further constraint on c_3 . However, if $\mathbf{A}_5^t C \mathbf{A}_6 \in \mathbb{Z} + \frac{1}{2}$, then we require that $c_3 = 0 \bmod 4$. In both cases, eqn (2.8) is satisfied without further constraint. Since c_3 is even, unimodularity requires that a_3 and d_3 are both odd, and the first term on the left-hand side of eqns (2.9) and (2.10) are both even. Then, to satisfy eqn (2.9) we only require that $c_3 \mathbf{A}_5^t C \mathbf{A}_6 = 0 \bmod 4$. If $\mathbf{A}_5^t C \mathbf{A}_6$ is even, this is already satisfied. However, if $\mathbf{A}_5^t C \mathbf{A}_6$ is odd, we now require that $c_3 = 0 \bmod 4$, whereas if $\mathbf{A}_5^t C \mathbf{A}_6 \in \mathbb{Z} + \frac{1}{2}$, then we require that $c_3 = 0 \bmod 8$. These are sufficient to ensure that eqn (2.10) is fully satisfied and, except in the third eventuality, that eqn (2.11) is too. In the case that $\mathbf{A}_5^t C \mathbf{A}_6 \in \mathbb{Z} + \frac{1}{2}$, this requires that $c_3 = 0 \bmod 16$. Thus, when \mathbf{A}_6 is *not* a lattice vector the conclusions are that (i) if $\mathbf{A}_5^t C \mathbf{A}_6$ is even, the modular group for T_3 is $\Gamma_0(2)$, (ii) if $\mathbf{A}_5^t C \mathbf{A}_6$ is odd, then the modular group is $\Gamma_0(4)$, but (iii) if $\mathbf{A}_5^t C \mathbf{A}_6 \in \mathbb{Z} + \frac{1}{2}$, then it is $\Gamma_0(16)$.

2.3 \mathbb{Z}_3 -planes

For the fixed plane T_1^2 that arises in the ω^k -twisted sector of the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold, the last \mathbb{Z}_3 -plane entry in Table 1, the constraint given in eqn (2.4) yields

$$\begin{pmatrix} 3\mathbf{A}_1^t C \mathbf{A}_1 & \mathbf{A}_1^t C \mathbf{A}_1 + 4\mathbf{A}_1^t C \mathbf{A}_2 + \mathbf{A}_2^t C \mathbf{A}_2 \\ 2\mathbf{A}_1^t C \mathbf{A}_2 - \mathbf{A}_1^t C \mathbf{A}_1 - \mathbf{A}_2^t C \mathbf{A}_2 & 3\mathbf{A}_2^t C \mathbf{A}_2 \end{pmatrix} \in 2\mathbb{Z} \quad (2.17)$$

where \mathbf{A}_1 and \mathbf{A}_2 differ by a lattice vector and $3\mathbf{A}_1$ is a lattice vector. As above, if \mathbf{A}_1 , and hence \mathbf{A}_2 , are lattice vectors, then this is automatically satisfied, and the analysis of the modular constraints is the same as for the \mathbb{Z}_4 -planes. The conclusions, in this case for T_1 , are the same.

We therefore need only consider the alternative, namely that \mathbf{A}_1 is *not* a lattice vector. Then to satisfy eqn (2.17) we must have that

$$3\mathbf{A}_1^t C \mathbf{A}_1 \in 2\mathbb{Z} \quad (2.18)$$

We may write

$$\mathbf{A}_2 - \mathbf{A}_1 := \mathbf{V} \quad (2.19)$$

where $\mathbf{V} \in \mathbb{Z}$ is a lattice vector, so that $\mathbf{V}^t C \mathbf{V} \in 2\mathbb{Z}$; it follows that $3\mathbf{A}_2^t C \mathbf{A}_2$ is also even, since $3\mathbf{A}_1^t C \mathbf{V}$ is an integer. Eqn (2.6) requires that $c_1 = 0 \bmod 3$. Then, the only further constraint from eqn (2.7) is that $c_1 \mathbf{A}_1^t C \mathbf{V} = 0 \bmod 2$. So, if $3\mathbf{A}_1^t C \mathbf{V} = 3\mathbf{A}_1^t C \mathbf{A}_2 \bmod 2$ is even, there is no further constraint on c_1 , but if $3\mathbf{A}_1^t C \mathbf{V}$ is odd, then we require that

c_1 is even, and hence $c_1 = 0 \bmod 6$. As before, eqn (2.8) is then satisfied without further constraint. In this case, unimodularity now means that we may choose that $a_1 = 1 \bmod 3 = d_1$, which ensures that the first term on the left-hand sides of eqns (2.9) and (2.10) are even. Then to satisfy them we require only that $c_1 \mathbf{A}_1^t C \mathbf{V} = 0 \bmod 6$. Thus, if $3 \mathbf{A}_1^t C \mathbf{V} = 0 \bmod 6$ there is no further constraint; if $3 \mathbf{A}_1^t C \mathbf{V} = 0 \bmod 2$ (but not $0 \bmod 6$), then $c_1 = 0 \bmod 9$. However if $3 \mathbf{A}_1^t C \mathbf{V}$ is odd, we require that $c_1 = 0 \bmod 18$. Finally, unimodularity requires that $2 - a_1 - d_1 = 0 \bmod c_1$, and to satisfy eqn (2.11) we require only that $c_1 (\mathbf{A}_1^t C \mathbf{V})^2 = 0 \bmod 4$. In the case that $3 \mathbf{A}_1^t C \mathbf{V}$ is even, there is no further constraint on $c_1 = 0 \bmod 9$. However, if $3 \mathbf{A}_1^t C \mathbf{V}$ is odd, we now require that $c_1 = 0 \bmod 36$. In the former case, the modular symmetry for T_1 is $\Gamma_0(9)$, and in the latter it is $\Gamma_0(36)$.

It is the availability of these large integers n in the reduced modular symmetry $\Gamma_0(n)$ of the \mathbb{Z}_8 and $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifolds that we believe may enable the gap between m_X and m_{string} to be bridged.

3 Gauge coupling constant unification

The question we shall now address is whether the inclusion of the moduli-dependent threshold corrections in the renormalisation group equations (1.1) allows unification of the Standard Model gauge coupling constants $g_a(\mu)$, ($a = 1, 2, 3$) at a scale $m_X \simeq 2 \times 10^{16}$ GeV consistently with the string scale $m_{\text{string}} \simeq 3.7 \times 10^{17}$ GeV. Taking all groups at level 1, this requires that for $a, b = 1, 2, 3$

$$\frac{m_{\text{string}}^2}{m_X^2} = \exp \left(- \frac{\Delta_a - \Delta_b}{b_a - b_b} \right) \quad (3.1)$$

and consistency requires that

$$\frac{\Delta_3 - \Delta_2}{\Delta_3 - \Delta_1} = \frac{b_3 - b_2}{b_3 - b_1} = \frac{5}{12} \quad (3.2)$$

in the Standard Model since

$$b_3 = -3, \quad b_2 = 1, \quad b_1 = \frac{33}{5} \quad (3.3)$$

Since $m_{\text{string}} > m_X$, we also require that

$$\Delta_3 - \Delta_2 > 0 \quad (3.4)$$

so that the exponent in eqn (3.1) is positive.

As we have already noted, the precise form of the threshold corrections in the presence of discrete Wilson lines is not known. However, since we know their modular symmetry, we may make plausible hypotheses of their form. In this paper we are only concerned with orbifolds in which there are no \mathbb{Z}_2 planes, in which case the complex structure moduli are fixed and only the modular symmetry of the Kähler moduli is relevant. In analogy with eqn (1.10) we assume² that Δ_a has the general form

$$\Delta_a = - \sum_{k=1}^3 (b_a'^k - \delta_{\text{GS}}^k) \ln[(T_k + \bar{T}_k) |\eta(p_k T_k) \eta(q_k T_k)|^2] \quad (3.5)$$

²The form assumed appears to be consistent with the results of [12].

where p_k and q_k are integers consistent with the modular symmetry in the $\mathcal{N} = 2$ plane T_k^2 . The coefficients $b'_a{}^k$ are given by the general formula [9]

$$b'_a{}^k = -c(G_a) + \sum_{\mathbf{R}_a} T(\mathbf{R}_a)(1 + 2n_{\mathbf{R}_a}^k) \quad (3.6)$$

Thus,

$$b'_3{}^k = 3 + \sum_{g=1}^3 (2n_{Q_g}^k + n_{u_g}^k + n_{d_g}^k) \quad (3.7)$$

$$b'_2{}^k = 5 + n_h^k + n_{\bar{h}}^k + \sum_{g=1}^3 (3n_{Q_g}^k + n_{L_g}^k) \quad (3.8)$$

$$b'_1{}^k = \frac{33}{5} + \frac{3}{5}(n_h^k + n_{\bar{h}}^k) + \frac{1}{5} \sum_{g=1}^3 (n_{Q_g}^k + 8n_{u_g}^k + 2n_{d_g}^k + 3n_{L_g}^k + 6n_{\ell_g}^k) \quad (3.9)$$

where $g = 1, 2, 3$ labels the three Standard Model fermion generations, with Q_g and L_g respectively the quark and lepton doublets, u_g, d_g and ℓ_g the quark and lepton singlets, and h, \bar{h} are the two Higgs doublets; n_X^k is the modular weight of the state X under the modular transformation given in eqn (1.4).

3.1 The \mathbb{Z}_8 -I orbifold

The point group generator θ has eigenvalues $e^{2\pi i v_k}$ with

$$(v_1, v_2, v_3) = \frac{1}{8}(1, -3, 2) \quad (3.10)$$

This may be realised by Casimirs acting on the $SO(9) \times SO(5)$ lattice or on the $SO(8)^{[2]} \times SO(5)$ lattice³, both of which have the form $T^4 \times T^2$. In both there is a single fixed $\mathcal{N} = 2$ plane T_3^2 in the θ^4 -twisted sector. Since θ acts as \mathbb{Z}_4 on this plane, only the Kähler modulus T_3 enters the threshold corrections. In this case eqns (3.4) and (3.2) require that

$$b'_3{}^3 - b'_2{}^3 > 0 \quad (3.11)$$

$$b'_3{}^3 - b'_1{}^3 > 0 \quad (3.12)$$

$$\frac{b'_3{}^3 - b'_2{}^3}{b'_3{}^3 - b'_1{}^3} = \frac{5}{12} \quad (3.13)$$

and eqns (3.7), .. (3.9) give

$$b'_3{}^3 - b'_2{}^3 = -2 - Y^3 + \sum_{g=1}^3 (-n_{Q_g}^3 + n_{u_g}^3) > 0 \quad (3.14)$$

$$\frac{5}{3}(b'_3{}^3 - b'_1{}^3) = -6 - Y^3 + \sum_{g=1}^3 (3n_{Q_g}^3 - n_{u_g}^3 - 2n_{\ell_g}^3) > 0 \quad (3.15)$$

³These two lattices are actually equivalent, as is shown in Table 5.4 of reference Fischer *et al.* [17]. We are grateful to the referee for drawing our attention to this paper.

State X	$-n_X^1$	$-n_X^2$	$-n_X^3$	$-\sum_k n_X^k$
Q, u, ℓ and L, h, \bar{h}, d	1	0	0	1
	0	1	0	1
	0	0	1	1
	$\frac{1}{2}$	$\frac{1}{2}$	0	1
	$\frac{7}{8}$	$\frac{3}{8}$	$\frac{3}{4}$	2
	$\frac{3}{8}$	$\frac{7}{8}$	$\frac{3}{4}$	2
	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	2
	$\frac{4}{8}$	$\frac{4}{8}$	$\frac{2}{4}$	
	$\frac{15}{8}$	$\frac{3}{8}$	$\frac{3}{4}$	3
	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{3}{4}$	3
Only L, h, \bar{h}, d	$\frac{7}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	3
	$\frac{3}{4}$	$\frac{7}{4}$	$\frac{1}{2}$	3
	$\frac{3}{4}$	$\frac{7}{4}$	$\frac{7}{4}$	3
	$\frac{8}{8}$	$\frac{3}{8}$	$\frac{7}{4}$	3
	$\frac{7}{8}$	$\frac{3}{8}$	$\frac{7}{4}$	3
	$\frac{8}{8}$	$\frac{8}{8}$	$\frac{4}{4}$	
	$\frac{23}{8}$	$\frac{3}{8}$	$\frac{3}{4}$	4
	$\frac{3}{8}$	$\frac{23}{8}$	$\frac{3}{4}$	4

Table 2. Modular weights of matter representations on the \mathbb{Z}_8 orbifold.

where

$$Y^k := n_h^k + n_{\bar{h}}^k - \sum_{g=1}^3 (n_{d_g}^k - n_{L_g}^k) \quad (k = 1, 2, 3) \quad (3.16)$$

Then, eliminating Y^3 (but not yet imposing the constraint in eqn (3.13)) gives

$$\frac{5}{3}(b'_3{}^3 - b'_1{}^3) - (b'_3{}^3 - b'_2{}^3) = -4 + 2 \sum_{g=1}^3 (2n_{Q_g}^3 - n_{u_g}^3 - n_{\ell_g}^3) \quad (3.17)$$

The calculation of the modular weights of the various states on this orbifold was treated in references [9, 18] and they are displayed in Table 2. It is easy to satisfy these constraints, including eqn (3.13), for several values of

$$b'_3{}^3 - b'_2{}^3 = \frac{1}{2}(1, 2, 3, 4) \quad (3.18)$$

The cancellation of target-space anomalies in T^4 requires also that $b'_a{}^k = \delta_{\text{GS}}^k$ independently of the gauge group a . Hence

$$b'_3{}^k = b'_2{}^k = b'_1{}^k \quad (k = 1, 2) \quad (3.19)$$

and

$$b'_3{}^k - b'_2{}^k = 0 = b'_3{}^k - b'_1{}^k \quad (k = 1, 2) \quad (3.20)$$

$-\sum_{k,g=1}^3 n_{Q_g}^k$	$-\sum_{k,g=1}^3 n_{u_g}^k$	$-\sum_{k,g=1}^3 n_{\ell_g}^k$	$-\sum_{k=1}^3 Y^k$
3	6	9	11
3	7	8	12
3	8	7	13
3	9	6	14
4	8	9	12
4	9	8	13

Table 3. Modular weights satisfying eqn (3.27) when $b'_3{}^3 - b'_2{}^3 = 2$.

Then summing over k gives

$$b'_3{}^3 - b'_2{}^3 = -6 - \sum_{k=1}^3 Y^k + \sum_{g,k=1}^3 (-n_{Q_g}^k + n_{u_g}^k) > 0 \quad (3.21)$$

$$\frac{5}{3}(b'_3{}^3 - b'_1{}^3) = -18 - \sum_{k=1}^3 Y^k + \sum_{g,k=1}^3 (3n_{Q_g}^k - n_{u_g}^k - 2n_{\ell_g}^k) > 0 \quad (3.22)$$

Now, it is clear from Table 2 that the sum $\sum_k n_X^k$ of the modular weights for any particle is an integer, and in fact this is generally true [9]. It then follows from these equations that

$$b'_3{}^3 - b'_2{}^3 \in \mathbb{Z} \quad (3.23)$$

$$\frac{5}{3}(b'_3{}^3 - b'_1{}^3) \in \mathbb{Z} \quad (3.24)$$

$$\frac{5}{3}(b'_3{}^3 - b'_1{}^3) - (b'_3{}^3 - b'_2{}^3) \in 2\mathbb{Z} \quad (3.25)$$

Evidently the half-integral solutions for $b'_3{}^3 - b'_2{}^3$ in eqn (3.18) are *not* consistent with the cancellation of modular anomalies. Further, if we *now* impose the constraint in eqn (3.13), it follows that

$$b'_3{}^3 - b'_2{}^3 \in 2\mathbb{Z} \quad (3.26)$$

so that only the earlier solutions in which $b'_3{}^3 - b'_2{}^3 = 2$ can be consistent with modular anomaly cancellation. Eliminating the $\sum_k Y^k$ terms from the above equations gives

$$6 + \frac{5}{6}(b'_3{}^3 - b'_1{}^3) - \frac{1}{2}(b'_3{}^3 - b'_2{}^3) = \sum_{g,k=1}^3 (2n_{Q_g}^k - n_{u_g}^k - n_{\ell_g}^k) \quad (3.27)$$

and the modular weights displayed in Table 2 allow only the solutions displayed in Table 3 when $b'_3{}^3 - b'_2{}^3 = 2$ (and $b'_3{}^3 - b'_1{}^3 = 24/5$). However, we also need to satisfy eqns (3.14) and (3.15) with $b'_3{}^3 - b'_2{}^3 = 2$. The only solutions are displayed in Table 4. The question is whether any of these is achieved by a set of modular weights that also yields one of the solutions given in Table 3. Three of these require $\sum_{g=1}^3 n_{u_g}^3 = -3$ or $\sum_{g=1}^3 n_{\ell_g}^3 = -3$. However this can only be achieved for u_g if $(n_{u_g}^1, n_{u_g}^2, n_{u_g}^3) = (0, 0, 1)$ for all three generations.

$-\sum_{g=1}^3 n_{Q_g}^3$	$-\sum_{g=1}^3 n_{u_g}^3$	$-\sum_{g=1}^3 n_{\ell_g}^3$	$-Y^3$
0	2	3	6
0	3	2	7
0	$\frac{9}{4}$	$\frac{11}{4}$	$\frac{25}{4}$
0	$\frac{11}{4}$	$\frac{9}{4}$	$\frac{27}{4}$
0	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{13}{2}$
$\frac{1}{2}$	3	3	$\frac{13}{2}$

Table 4. Modular weights satisfying eqns (3.14) and (3.15) when $b_3'^3 - b_2'^3 = 2$.

But then $\sum_{k,g=1}^3 n_{u_g}^k = -3$ which is not allowed by any of the solutions given in Table 3. The same argument applies to ℓ_g . Similarly, there are two solutions in which $\sum_{g=1}^3 n_{u_g}^3 = -11/4$ or $\sum_{g=1}^3 n_{\ell_g}^3 = -11/4$, and the allowed sets of modular weights satisfying this have $\sum_{k,g=1}^3 n_{u_g}^k = -4, -5$, neither of which is allowed by the solutions in Table 3. It follows that the only possibility is the solution with $\sum_{g=1}^3 n_{u_g}^3 = -5/2 = \sum_{g=1}^3 n_{\ell_g}^3$, which can arise from modular weights with $\sum_{k,g=1}^3 n_{u_g}^k = -6, -7 = \sum_{k,g=1}^3 n_{\ell_g}^k$. Unfortunately none of the solutions in Table 3 have *both* $\sum_{k,g=1}^3 n_{u_g}^k = -6, -7$ and $\sum_{k,g=1}^3 n_{\ell_g}^k = -6, -7$. The conclusion is that, independently of the moduli-dependence of the threshold corrections, it is impossible to satisfy the unification constraint given in eqn (3.13) consistently with the cancellation of modular anomalies.

In their treatment of this question, instead of imposing eqn (3.13), Ibáñez and Lüst [9] allowed a range of values of

$$r := \frac{\frac{5}{3}(b_3'^3 - b_1'^3)}{b_3'^3 - b_2'^3} \quad (3.28)$$

They required that the ratio

$$\gamma := \frac{1+r}{\frac{3}{5}r-1} \quad (3.29)$$

is in the range

$$2.7 \leq \gamma \leq 3.7 \quad (3.30)$$

corresponding to

$$3.88 \leq r \leq 5.95 \quad (3.31)$$

which includes the value $r = 4$ corresponding to eqn (3.13). Eqns (3.23) ... (3.25) hold quite generally, and if we choose

$$b_3'^3 - b_2'^3 = 1 \quad (3.32)$$

then the above range only allows $r = 5$ consistently with eqn (3.25), in which case

$$b_3'^3 - b_1'^3 = 3 \quad (3.33)$$

In this case the solutions of eqns (3.21) and (3.22) are given in Table 5, and the solutions of eqns (3.14) and (3.15) in Table 6. The solution given in [9] corresponds to the third

$-\sum_{i,g=1}^3 n_{Q_g}^i$	$-\sum_{i,g=1}^3 n_{u_g}^i$	$-\sum_{i,g=1}^3 n_{\ell_g}^i$	$-\sum_{i=1}^3 Y^i$
3	5	9	9
3	6	8	10
3	7	7	11
3	8	6	12
3	9	5	13
4	7	9	10
4	8	8	11
4	9	7	12
5	9	9	11

Table 5. Modular weights satisfying eqns (3.21) and (3.22) when $b'_3 - b'_2 = 1$.

$-\sum_{g=1}^3 n_{Q_g}^3$	$-\sum_{g=1}^3 n_{u_g}^3$	$-\sum_{g=1}^3 n_{\ell_g}^3$	$-Y^3$
0	1	3	4
0	$\frac{5}{4}$	$\frac{11}{4}$	$\frac{17}{4}$
0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{9}{2}$
0	$\frac{7}{4}$	$\frac{9}{4}$	$\frac{19}{4}$
0	2	2	5
0	$\frac{9}{4}$	$\frac{7}{4}$	$\frac{21}{4}$
0	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{11}{2}$
0	$\frac{11}{4}$	$\frac{5}{4}$	$\frac{23}{4}$
0	3	1	6
$\frac{1}{2}$	2	3	$\frac{9}{2}$
$\frac{1}{2}$	$\frac{9}{4}$	$\frac{11}{4}$	$\frac{19}{4}$
$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	5
$\frac{1}{2}$	$\frac{11}{4}$	$\frac{9}{4}$	$\frac{21}{4}$
$\frac{1}{2}$	3	2	$\frac{11}{2}$
$\frac{3}{4}$	$\frac{5}{2}$	3	$\frac{19}{4}$
$\frac{3}{4}$	$\frac{11}{4}$	$\frac{11}{4}$	5
$\frac{3}{4}$	3	$\frac{5}{2}$	$\frac{21}{4}$
1	3	3	5

Table 6. Modular weights satisfying eqns (3.14) and (3.15) when $b'_3 - b'_2 = 1$.

line in each Table, and it is easily verified that their solution satisfies the modular anomaly cancellation constraints given in eqn (3.19). Thus the \mathbb{Z}_8 -I orbifold does allow gauge coupling constant unification if the weaker constraint given in eqn (3.31) is imposed rather than $r = 4$ that is required by eqn (3.13). The outstanding question then is whether this can be achieved with a reasonable value of the Kähler modulus T_3 of order unity. We shall address this question in §4.

3.2 The $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold

The point group generators θ, ω have eigenvalues $e^{2\pi i v_k}$ with

$$(v_1, v_2, v_3)_\theta = \frac{1}{3}(0, -1, 1) \quad (3.34)$$

$$(v_1, v_2, v_3)_\omega = \frac{1}{6}(1, -1, 0) \quad (3.35)$$

which may be realised using Casimirs on the $SU(3)^3$ lattice. It is convenient to reorder the twists here compared with those used in [13] and Table 1 of §2, so that now T_3^2 is the \mathbb{Z}_3 -plane, while $T_{1,2}^2$ are both \mathbb{Z}_6 -planes. Evidently all three planes $T_{1,2,3}^2$ are $\mathcal{N} = 2$ planes, so all three Kähler moduli $T_{1,2,3}$ contribute to the threshold corrections. The complex structure moduli $U_{1,2,3} = e^{2\pi i/3}$ are all fixed, since none of them is a \mathbb{Z}_2 -plane. In this case eqns (3.4) and (3.2) require that

$$\sum_{k=1}^3 (b_3'^k - b_2'^k) \ln \alpha_k < 0 \quad (3.36)$$

$$\sum_{k=1}^3 (b_3'^k - b_1'^k) \ln \alpha_k < 0 \quad (3.37)$$

$$\frac{\sum_{k=1}^3 (b_3'^k - b_2'^k) \ln \alpha_k}{\sum_{k=1}^3 (b_3'^k - b_1'^k) \ln \alpha_k} = \frac{5}{12} \quad (3.38)$$

where

$$\alpha_k := (T_k + \bar{T}_k) |\eta(p_k T_k) \eta(q_k T_k)|^2 \quad (3.39)$$

The quantities $\ln \alpha_k$ are dominated the contribution from the Dedekind eta functions whose logarithm is negative in the region of interest, as can be seen in eqn (4.3); this explains the difference between the inequalities in eqns (3.36) and (3.37) compared with those in eqns (3.11) and (3.12). If a single modulus, T_3 say, dominates, these equations reduce to the constraints given previously in eqns (3.14), (3.15) and (3.13). Further, there are no modular anomalies on the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold. In this case the modular weights [9, 18] are given in Table 7, and using these it is clear from eqn (3.14) that $3(b_3'^3 - b_2'^3) \in \mathbb{Z}$. In fact, using eqn (3.13), it follows from eqn (3.17) that $3(b_3'^3 - b_2'^3) \in 2\mathbb{Z}$. Table 8 lists all solutions with $b_3'^3 - b_2'^3 > 0$, and shows that the only allowed values are $3(b_3'^3 - b_2'^3) = 2, 4, 6, 8$; it is easy to see that there are many allowed choices of the modular weights $n_{(h, \bar{h}, d, L)_g}^3$ that yield the values of Y^3 that arise.

However, it is probably unrealistic to assume that a single modulus dominates on this orbifold. Instead we assume that

$$T_1 = T_2 = T_3 := T \quad (3.40)$$

We wish to exploit the considerably reduced modular symmetry that may be associated with the Kähler modulus T_3 when there are Wilson lines in this \mathbb{Z}_3 plane. Thus we choose $\alpha_3 \neq \alpha_1 = \alpha_2$; $T_{1,2}^2$ are both \mathbb{Z}_6 planes, so it is perfectly feasible that their modular

symmetries are the same (even if there are non-trivial Wilson lines) and that $\alpha_1 = \alpha_2$. Then eqn (3.38) gives

$$\frac{\sum_{k=1}^2 (b_3'^k - b_2'^k) \ln \alpha_1 + (b_3'^3 - b_2'^3) \ln \alpha_3}{\sum_{k=1}^2 (b_3'^k - b_1'^k) \ln \alpha_1 + (b_3'^3 - b_2'^3) \ln \alpha_3} = \frac{5}{12} \quad (3.41)$$

where

$$\alpha_1 = \alpha_2 = (T + \bar{T}) |\eta(p_1 T) \eta(q_1 T)|^2 \quad (3.42)$$

This is satisfied independently of α_1 and α_3 if,

$$\frac{\sum_{k=1}^2 (b_3'^k - b_2'^k)}{\sum_{k=1}^2 (b_3'^k - b_1'^k)} = \frac{5}{12} = \frac{b_3'^3 - b_2'^3}{b_3'^3 - b_1'^3} \quad (3.43)$$

and this implies that

$$\frac{\sum_{k=1}^3 (b_3'^k - b_2'^k)}{\sum_{k=1}^3 (b_3'^k - b_1'^k)} = \frac{5}{12} \quad (3.44)$$

We have just given (in Table 8) all solutions of the second equation in equation (3.43). So the question is whether there are solutions of eqn (3.44) that are achievable with the same sets of modular weights as are needed to solve the former.

Summing over $k = 1, 2, 3$, it follows, as in eqns (3.21) and (3.22), that

$$\sum_{k=1}^3 (b_3'^k - b_2'^k) = -6 - \sum_{k=1}^3 Y^k + \sum_{g,k=1}^3 (-n_{Q_g}^k + n_{u_g}^k) \quad (3.45)$$

$$\frac{5}{3} \sum_{k=1}^3 (b_3'^k - b_1'^k) = -18 - \sum_{k=1}^3 Y^k + \sum_{g,k=1}^3 (3n_{Q_g}^k - n_{u_g}^k - 2n_{\ell_g}^k) \quad (3.46)$$

where Y_k is given in eqn (3.16). As before, we may infer, quite generally, that

$$\sum_{k=1}^3 (b_3'^k - b_2'^k) \in \mathbb{Z} \quad (3.47)$$

$$\frac{5}{3} \sum_{k=1}^3 (b_3'^k - b_1'^k) \in \mathbb{Z} \quad (3.48)$$

$$\frac{5}{3} \sum_{k=1}^3 (b_3'^k - b_1'^k) - \sum_{k=1}^3 (b_3'^k - b_2'^k) = 0 \pmod{2} \quad (3.49)$$

Imposing eqn (3.44) then gives

$$3 \sum_{k=1}^3 (b_3'^k - b_2'^k) = -12 + \sum_{g,k=1}^3 (4n_{Q_g}^k - 2n_{u_g}^k - 2n_{\ell}^k) \quad (3.50)$$

Thus, in this case

$$\sum_{k=1}^3 (b_3'^k - b_2'^k) = 0 \pmod{2} \quad (3.51)$$

$$\sum_{g,k=1}^3 (n_{Q_g}^k + n_{u_g}^k + n_{\ell}^k) = 0 \pmod{3} \quad (3.52)$$

and using the modular weights in Table 7 the only solutions are given in Table 9. As before, there are many sets of modular weights that give rise to the values found of $\sum_{k=1}^3 Y_k$.

We first need to know whether any of the solutions displayed in Tables 8 and 9 coincide. Consider solution IV of Table 9. From Table 7, we see that $\sum_{g,k=1}^3 n_{Q_g}^k = 0$ requires that $n_{Q_g}^3 = 0$ for all g , so this line can only be consistent with some or all of the four lines in Table 8 in which $\sum_{g=1}^3 n_{Q_g}^3 = 0$. It also follows from Table 7 that $-\sum_{g,k=1}^3 n_{u_g}^k = 6$ can only be realised with $-\sum_{k=1}^3 n_{u_g}^k = 2$ for all g , and this requires that $-3n_{u_g}^3 = (0, 1, 2)$ so that $-\sum_{g=1}^3 3n_{u_g}^3 = (0, 1, 2, \dots, 6)$. Similarly for $-\sum_{g=1}^3 3n_{\ell_g}^3$. Consider the solutions in the first line of Table 8 with $b'_3{}^3 - b'_2{}^3 = 2/3$. They have the property that $-\sum_g 3n_{u_g}^3 - \sum_g 3n_{\ell_g}^3 = 9$. Thus only the four combinations $(-\sum_g 3n_{u_g}^3, -\sum_g 3n_{\ell_g}^3) = ((3, 4, 5, 6), (6, 5, 4, 3))$ can be realised in this case. The values of $3Y_3$ corresponding to these four solutions are $3Y_3 = (10, 11, 12, 13)$. Likewise, only one of the solutions in the line with $b'_3{}^3 - b'_2{}^3 = 4/3$ may be realised, namely the one with $(-\sum_g 3n_{u_g}^3, -\sum_g 3n_{\ell_g}^3) = (6, 6)$. The solutions with $b'_3{}^3 - b'_2{}^3 = 2$ and $8/3$ cannot be realised from solution IV. In summary, using solution IV we can realise only the solutions in Table 8 in which

$$\left(b'_3{}^3 - b'_2{}^3, -\sum_g 3n_{Q_g}^3, -\sum_g 3n_{u_g}^3, -\sum_g 3n_{\ell_g}^3, 3Y_3 \right) = \left(\frac{2}{3}, 0, (3, 4, 5, 6), (6, 5, 4, 3), (10, 11, 12, 13) \right) \quad (3.53)$$

$$= \left(\frac{4}{3}, 0, 6, 6, 16 \right) \quad (3.54)$$

Also, as above, solutions Ia, Ib, IIa, IIb of Table 9 can only be consistent with some or all of the four lines in Table 8 in which $\sum_{g=1}^3 n_{Q_g}^3 = 0$. Proceeding as above, for solution Ia, we may infer that for two generations, $g = 1, 2$ say, $-\sum_{k=1}^3 n_{u_g}^k = 2$, whereas for the third $-\sum_{k=1}^3 n_{u_3}^k = 1$. In this last case, $-3n_{u_3}^3 = (0, 1, 2, 3)$, so that for solution Ia $-\sum_{g=1}^3 3n_{u_g}^3 = (0, 1, 2, \dots, 7)$. It follows similarly that $-\sum_{g=1}^3 3n_{\ell_g}^3 = (0, 1, 2, \dots, 8)$; for solution Ib, the ranges for u and ℓ are interchanged. Thus now we can realise the solutions in Table 8 in which

$$\left(b'_3{}^3 - b'_2{}^3, -\sum_g 3n_{Q_g}^3, -\sum_g 3n_{u_g}^3, -\sum_g 3n_{\ell_g}^3, 3Y_3 \right) = \left(\frac{2}{3}, 0, (1, 2, \dots, 8), (8, 7, \dots, 1), (9, 10, \dots, 16) \right) \quad (3.55)$$

$$= \left(\frac{4}{3}, 0, (5, 6, 7), (7, 6, 5), 16 \right) \quad (3.56)$$

$$= (2, 0, (7, 8), (8, 7), (19, 20)) \quad (3.57)$$

Proceeding in this way, we find that the solutions in Table 9 generate some of the solutions in Table 8, but not all. Which ones are displayed in Table 10, and for comparison in the last column of Table 8.

Finally, we need to check that the consistent solutions may all be realised by consistent sets of modular weights for h, \bar{h}, d_g, L_g . For example, consider solution Ia in Table 9. The particular choice of the modular weights displayed in Table 11 realises the values

(2/3, 0, 4, 5, 12) on the right-hand side of eqn (3.55) and satisfies all of our constraints. The other solutions may be similarly realised.

4 Estimation of the Kähler modulus needed for unification

It follows from eqn (3.1) that

$$-2(b_3 - b_2) \ln \frac{m_{\text{string}}}{m_X} = \Delta_3 - \Delta_2 \quad (4.1)$$

$$= -\sum_{k=1}^3 (b_3'^k - b_2'^k) \ln[(T_k + \bar{T}_k) |\eta(p_k T_k) \eta(q_k T_k)|^2] \quad (4.2)$$

For all values of interest we may approximate

$$\ln(\eta(T)) \simeq -\frac{\pi T}{12} \quad (4.3)$$

Thus, since $b_3 - b_2 = -4$ and $\ln(m_{\text{string}}/m_X) \simeq 2.92$, in general, we seek allowed values of $p_k + q_k$ for which there are solutions of

$$23.3 \simeq \sum_{k=1}^3 (b_3'^k - b_2'^k) \left[\frac{\pi(p_k + q_k)T_k}{6} - \ln(2T_k) \right] \quad (4.4)$$

with $T_k \sim 1$.

4.1 Unification in the \mathbb{Z}_8 -I orbifold

In the case of the \mathbb{Z}_8 -I orbifold the threshold corrections are determined entirely by the modulus T_3 . As discussed in §3.1, it is impossible to satisfy the “strong” unification constraint given in eqn (3.13). However if instead, as in [9], we impose the weaker constraint given in eqn (3.31), then there is an allowed solution in which $b_3'^3 - b_2'^3 = 1$, corresponding to

$$\frac{b_3'^3 - b_2'^3}{b_3'^3 - b_1'^3} = \frac{4}{12} \quad (4.5)$$

In this case, then, we must satisfy

$$23.3 \simeq \frac{\pi(p_3 + q_3)T_3}{6} - \ln(2T_3) \quad (4.6)$$

In §2.2 we showed that, depending on the choice of Wilson lines, the modular symmetry associated with a \mathbb{Z}_4 -plane may be $\Gamma_0(2), \Gamma_0(4)$ or $\Gamma_0(16)$. If we choose Wilson lines \mathbf{A}_5 and \mathbf{A}_6 such that $\mathbf{A}_5^t C \mathbf{A}_6 \in \mathbb{Z} + \frac{1}{2}$, it is *plausible* that the threshold corrections involve terms with

$$p_3 = 16 = q_3 \quad (4.7)$$

in which case eqn (4.6) is satisfied when

$$T_3 \simeq 1.5 \quad (4.8)$$

4.2 Unification in the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold

As discussed in §3.2, all three Kähler moduli contribute to the threshold corrections in this case. However, if the modulus T_3 associated with the \mathbb{Z}_3 -plane dominates, then it is certainly possible to satisfy the strong unification constraint. The allowed values of $b'_3{}^3 - b'_2{}^3$, given in Table 8, are

$$b'_3{}^3 - b'_2{}^3 = \frac{1}{3}(2, 4, 6, 8) \quad (4.9)$$

If we choose Wilson lines \mathbf{A}_5 and \mathbf{A}_6 in T_3^2 that are not lattice vectors such that $3\mathbf{A}_5^t C \mathbf{A}_6 \in 2\mathbb{Z}$, then the modular group associated with the \mathbb{Z}_3 -plane is $\Gamma_0(9)$, whereas if $3\mathbf{A}_5^t C \mathbf{A}_6 \in 2\mathbb{Z} + 1$, then it is $\Gamma_0(36)$. Table 12 shows how it is possible to satisfy the strong unification constraint with $T_3 \sim 1$ for all allowed values of $b'_3{}^3 - b'_2{}^3$ by a judicious, but plausible, choice of p_3 and q_3 .

It is more natural to assume that all three moduli contribute and are of the same size $T_1 \simeq T_2 \simeq T_3 = T$, and we have shown in Table 10 that the strong unification condition can still be satisfied, but only for the combinations of $b'_3{}^3 - b'_2{}^3$ and $\sum_{k=1}^3 (b'_3{}^k - b'_2{}^k)$ given in the first and third columns of Table 13. Wilson lines \mathbf{A}_{2k-1} and \mathbf{A}_{2k} in the \mathbb{Z}_6 -planes T_k^2 ($k = 1, 2$) are always $E_8 \times E_8$ lattice vectors. If $\mathbf{A}_{2k-1}^t C \mathbf{A}_{2k} \in 2\mathbb{Z}$, then the associated modular symmetry is the full $PSL(2, \mathbb{Z})$. Alternatively, if $\mathbf{A}_{2k-1}^t C \mathbf{A}_{2k} \in 2\mathbb{Z} + 1$, then it is $\Gamma_0(4)$. We choose the Wilson lines such that the symmetry is the same in both \mathbb{Z}_6 -planes, and $\alpha_1 = \alpha_2$. Then eqn (4.4) becomes

$$23.3 \simeq \frac{\pi T}{6} \left[(b'_3{}^3 - b'_2{}^3)(p_3 + q_3) + \sum_{k=1}^2 (b'_3{}^k - b'_2{}^k)(p_1 + q_1) \right] - \ln(2T) \sum_{k=1}^3 (b'_3{}^k - b'_2{}^k) \quad (4.10)$$

Table 13 shows how it is possible to satisfy the strong unification constraint with $T \sim 1$ for all allowed values of $(b'_3{}^3 - b'_2{}^3, \sum_{k=1}^3 (b'_3{}^k - b'_2{}^k))$ by a judicious, but plausible, choice of p_3, q_3, p_1 and q_1 . (The solution in the last line is the same as the solution in the third line of Table 12 because the moduli associated with T_1^2 and T_2^2 do not contribute; in other words, the threshold corrections are dominated by T_3 in this case.)

5 Conclusions

Our main result is that for certain classes of discrete Wilson lines the (Kähler) modular symmetry $\Gamma_0(n)$ associated with fixed \mathbb{Z}_4 - and \mathbb{Z}_3 -planes is considerably less ($n = 16$ and 36 respectively) than was previously realised. Such planes arise in the \mathbb{Z}_8 and $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifolds respectively. Both orbifolds may be realised by Casimirs acting on $T^4 \oplus T^2$ lattices, with T^2 the $\mathcal{N} = 2$ fixed plane. Neither has any fixed \mathbb{Z}_2 -planes, so the only unfixed moduli are Kähler. We also showed that the modular symmetry for \mathbb{Z}_6 planes may be reduced to $\Gamma_0(4)$ for certain classes of discrete Wilson lines.

For the \mathbb{Z}_8 case, the allowed modular weights of the Standard Model matter content make it impossible to satisfy the “strong” unification constraint for the supersymmetric Standard Model gauge coupling constants given in eqn (3.13) consistently with the cancellation of modular anomalies. It is, however, possible to satisfy a somewhat weaker constraint

that is compatible with the measured values of the gauge coupling strengths $g_a(\mu)$ at low scales $\mu \sim m_Z$. In this case the threshold corrections Δ_a depend only on the single modulus T_3 . Using a conjectured form of Δ_a that possesses the reduced modular symmetry $\Gamma_0(16)$, we have verified that the gap between the observed unification scale $m_X \sim 2 \times 10^{16}$ GeV and the string scale $m_{\text{string}} \simeq 3.7 \times 10^{17}$ GeV can now be bridged with a plausible value of $T_3 \sim 1.5$.

For the $\mathbb{Z}_3 \times \mathbb{Z}_6$ case, the situation is more complicated because the orbifold is realised on a $T_1^2 \oplus T_2^2 \oplus T_3^2$ lattice and all three tori are $\mathcal{N} = 2$ fixed planes. Thus, all three Kähler moduli $T_{1,2,3}$ are unfixed and the threshold corrections depend upon all three; T_1^2 and T_2^2 are fixed \mathbb{Z}_6 -planes, while T_3^2 is a fixed \mathbb{Z}_3 -plane. Since there is no reason why any one of the three moduli should dominate, we assume that all three have the same value T . In this case, we find that the allowed modular weights *are* compatible with the strong unification constraint (for several different values of the allowed anomaly coefficients b_a^k). Further, using the conjectured form of the threshold corrections that possesses the full $PSL(2, \mathbb{Z})$ modular symmetry for $T_{1,2}$ and $\Gamma_0(36)$ for T_3 , we showed that the gap between m_X and m_{string} can be bridged in this case too with a plausible value of $T \sim 0.9$. Alternatively, if, notwithstanding our prejudice, a highly anisotropic orbifold is realised with T_3 the dominant Kähler modulus, then again we find that the allowed modular weights are compatible with strong unification also for several different values of the coefficients b_a^k . Using the conjectured form of the threshold corrections with modular symmetry $\Gamma_0(9)$ and/or $\Gamma_0(36)$ in this case allows the gap to be bridged with values of T_3 in the range $0.94 - 1.34$.

Our results suggest, but do not prove, that the considerably reduced modular symmetry that arises for large classes of discrete Wilson line backgrounds on the \mathbb{Z}_8 and $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifolds may well allow the observed unification of the gauge coupling strengths to occur consistently with the scale naturally arising from the heterotic string and with plausible values of the Kähler moduli that determine the overall scale of these orbifolds. Determining if this suggestion *is* true would require the prior calculation of the threshold corrections in the presence of discrete Wilson lines. Since these Wilson lines mix winding numbers with internal lattice momenta, such a calculation is considerably more complicated than when they are absent.

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State X	$-n_X^1$	$-n_X^2$	$-n_X^3$	$-\sum_i n_X^i$
Q, u, ℓ and L, h, \bar{h}, d	$\frac{5}{6}$ $-\frac{5}{6}$	$-\frac{5}{6}$ $\frac{5}{6}$	0 0	0 0
	1 0 0 $\frac{5}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{2}$ 0 0 $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{3}$	0 1 0 $\frac{1}{6}$ $\frac{5}{6}$ $\frac{2}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{1}{3}$ 0	0 0 1 0 0 0 0 0 0 $\frac{2}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{3}$ $\frac{1}{3}$	1 1 1 1 1 1 1 1 1 1 1 1 1 1
	$\frac{1}{6}$ $\frac{11}{6}$ $\frac{1}{6}$ $\frac{5}{6}$ $\frac{5}{6}$ $\frac{5}{6}$ $\frac{2}{3}$ $\frac{1}{3}$ $\frac{1}{2}$	$\frac{11}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{5}{6}$ $\frac{1}{6}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{1}{6}$	0 0 $\frac{1}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$	2 2 2 2 2 2 2 2 2
Only L, h, \bar{h}, d	$-\frac{11}{6}$ $-\frac{11}{6}$	$\frac{5}{6}$ $\frac{5}{6}$	0 0	-1 -1
	$-\frac{1}{3}$ $\frac{1}{3}$ 0 0 $-\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{3}$ $\frac{1}{3}$ $-\frac{1}{3}$ $\frac{1}{3}$ 0 0	0 0 $\frac{1}{3}$ $-\frac{1}{3}$ $\frac{1}{3}$ $-\frac{1}{3}$	0 0 0 0 0 0
	$\frac{1}{3}$ $\frac{5}{3}$ $\frac{1}{3}$ 0 0 $\frac{1}{3}$ $\frac{5}{3}$ $\frac{1}{3}$	$\frac{5}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{5}{3}$ 0 $\frac{5}{3}$ 0	0 0 $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$	2 2 2 2 2 2 2 2
	$\frac{11}{6}$ $\frac{5}{6}$ $\frac{11}{6}$ $\frac{1}{6}$ $\frac{1}{2}$	$\frac{5}{6}$ $\frac{11}{6}$ $\frac{1}{2}$ $\frac{11}{6}$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{1}{3}$	3 3 3 3 3
	$\frac{17}{6}$ $\frac{5}{6}$ $\frac{11}{6}$ $\frac{5}{6}$ $\frac{1}{3}$	$\frac{5}{6}$ $\frac{17}{6}$ $\frac{11}{6}$ $\frac{5}{6}$ $\frac{1}{3}$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$	4 4 4 4 4

Table 7. Modular weights of matter representations on the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold.

$b'_{32}{}^3$	$-\sum_g 3n_{Q_g}^3$	$-\sum_g 3n_{u_g}^3$	$-\sum_g 3n_{\ell_g}^3$	$-3Y^3$	Table 9 solution
$\frac{2}{3}$	0	(0, 1, ..., 9)	(9, 8, ..., 0)	(8, 9, ..., 17)	=II \supset I \supset III \supset IV
	1	(2, 3, ..., 9)	(9, 8, ..., 2)	(9, 10, ..., 16)	\supset III
	2	(4, 5, ..., 9)	(9, 8, ..., 4)	(10, 11, ..., 15)	\supset III
	3	(6, 7, 8, 9)	(9, 8, 7, 6)	(11, 12, 13, 14)	
	4	(8, 9)	(9, 8)	(12, 13)	
$\frac{4}{3}$	0	(3, 4, ..., 9)	(9, 8, ..., 3)	(13, 14, ..., 19)	=II \supset I \supset III \supset IV
	1	(5, 6, ..., 9)	(9, 8, ..., 5)	(14, 15, ..., 18)	
	2	(7, 8, 9)	(9, 8, 7)	(15, 16, 17)	
	3	9	9	16	
2	0	(6, 7, 8, 9)	(9, 8, 7, 6)	(18, 19, 20, 21)	= II \supset I
	1	(8, 9)	(9, 8)	(19, 20)	
$\frac{8}{3}$	0	9	9	23	

Table 8. All solutions of eqn (3.17) with $b'_3{}^3 - b'_2{}^3 > 0$. The last column indicates which solutions in Table 9 wholly or partially realise this solution here.

Solution	$\sum_{k=1}^3 (b'_3{}^k - b'_2{}^k)$	$-\sum_{g,k=1}^3 n_{Q_g}^k$	$-\sum_{g,k=1}^3 n_{u_g}^k$	$-\sum_{g,k=1}^3 n_{\ell_g}^k$	$\sum_{k=1}^3 Y_k$
Ia	2	0	5	4	13
Ib	2	0	4	5	12
IIa	2	0	6	3	14
IIb	2	0	3	6	11
IIIa	2	1	6	5	13
IIIb	2	1	5	6	12
IV	4	0	6	6	16

Table 9. All solutions of eqn (3.50) with $\sum_{k=1}^3 (b'_3{}^k - b'_2{}^k) > 0$.

Solution	$b_3^k - b_2^3$	$-\sum_g 3n_{Q_g}^3$	$-\sum_g 3n_{u_g}^3$	$-\sum_g 3n_{\ell_g}^3$	$-3Y^3$
I	$\frac{2}{3}$	0	(1, 2, ..., 8)	(8, 7, ..., 1)	(9, 10, ..., 16)
	$\frac{4}{3}$	0	(4, 5, ..., 8)	(8, 7, ..., 4)	(14, 15, ..., 18)
	2	0	(7, 8)	(8, 7)	(19, 20)
II	$\frac{2}{3}$	0	(0, 1, ..., 9)	(9, 8, ..., 0)	(8, 9, ..., 17)
	$\frac{4}{3}$	0	(3, 4, ..., 9)	(9, 8, ..., 3)	(13, 14, ..., 19)
	2	0	(6, 7, 8, 9)	(9, 8, 7, 6)	(18, 19, 20, 21)
III	$\frac{2}{3}$	0	(2, 3, ..., 7)	(7, 6, ..., 2)	(10, 11, ..., 15)
	$\frac{4}{3}$	0	(5, 6, 7)	(7, 6, 5)	(15, 16, 17)
	$\frac{2}{3}$	1	(4, 5, 6, 7)	(7, 6, 5, 4)	(11, 12, 13, 14)
	$\frac{2}{3}$	2	(6, 7)	(7, 6)	(12, 13)
IV	$\frac{2}{3}$	0	(3, 4, 5, 6)	(6, 5, 4, 3)	(12, 13, 14, 15)
	$\frac{4}{3}$	0	6	6	16

Table 10. Consistent solutions from Table 9.

Particle X	$-n_X^1$	$-n_X^2$	$-n_X^3$	$-\sum_k n_X^k$
h	$\frac{11}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	3
\bar{h}	$\frac{11}{6}$	$\frac{1}{2}$	$\frac{2}{3}$	3
L_1	0	$\frac{1}{3}$	$\frac{5}{3}$	2
L_2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	2
L_3	$\frac{11}{6}$	$\frac{1}{2}$	$\frac{2}{3}$	3
d_1	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
d_2	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
d_3	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
	$-Y^1 = \frac{11}{3}$	$-Y^2 = \frac{16}{3}$	$-Y^3 = 4$	$-\sum_k Y^k = 13$
Q_1	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
Q_2	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
Q_3	$\frac{5}{6}$	$-\frac{5}{6}$	0	0
u_1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	2
u_2	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	2
u_3	$\frac{1}{3}$	0	$\frac{2}{3}$	1
ℓ_1	0	$\frac{2}{3}$	$\frac{1}{3}$	1
ℓ_2	0	$\frac{1}{3}$	$\frac{2}{3}$	1
ℓ_3	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1
	$b_{32}^1 = \frac{13}{6}$ $b_{31}^1 = -\frac{39}{10}$	$b_{32}^2 = -\frac{5}{6}$ $b_{31}^2 = \frac{71}{10}$	$b_{32}^3 = \frac{2}{3}$ $b_{31}^3 = \frac{8}{5}$	$\sum_k b_{32}^k = 2$ $\sum_k b_{31}^k = \frac{24}{5}$

Table 11. Consistent modular weights for solution Ia.

$b_3'^3 - b_2'^3$	p_3	q_3	T_3
$\frac{2}{3}$	36	36	0.94
$\frac{4}{3}$	36	1	0.94
2	9	9	1.34
$\frac{8}{3}$	9	9	1.0

Table 12. Choices of p_3 and q_3 for the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold that allow unification with $T_3 \sim 1$.

$b_3'^3 - b_2'^3$	$\sum_{k=1}^2 (b_3'^k - b_2'^k)$	$\sum_{k=1}^3 (b_3'^3 - b_2'^k)$	p_3	q_3	p_1	q_1	T
$\frac{2}{3}$	$\frac{4}{3}$	2	36	36	1	1	0.92
$\frac{2}{3}$	$\frac{10}{3}$	4	36	36	1	1	0.9
$\frac{4}{3}$	$\frac{2}{3}$	2	36	1	1	1	0.92
$\frac{4}{3}$	$\frac{8}{3}$	4	36	1	1	1	0.9
2	0	2	9	9	-	-	1.34

Table 13. Choices of p_3, q_3, p_1 and q_1 for the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifold that allow unification with $T \sim 1$.